A NOTE ON THE FOURIER COEFFICIENTS OF HALF-INTEGRAL WEIGHT MODULAR FORMS

NARASIMHA KUMAR AND SOMA PURKAIT

Abstract. In this note, we show that the algebraicity of the Fourier coefficients of half-integral weight modular forms can be determined by checking the algebraicity of the first "few" of them. We also give a necessary and sufficient condition for a half-integral weight modular form to be in the Kohnen's +-subspace.

1. Introduction

In the theory of modular forms, it is of fundamental interest to understand the algebraicity of the Fourier coefficients of modular forms. It is well-known that a normalized Hecke eigenform of integral weight has algebraic Fourier coefficients, since the Hecke eigenvalues coincide with the Fourier coefficients. Moreover, there exists a number field containing all these Fourier coefficients. However, for half-integral weight modular forms, we are not aware of any such results.

In this note, we show that if the Fourier coefficients of a half-integral weight modular form are algebraic up to the Sturm's bound (for the half-integral weight modular forms), which is specified in terms of the level and weight of the corresponding modular form, then so are all others (cf. Theorem 3.3 in the text). In the last section, we also give a necessary and sufficient condition, in terms of Sturm's bound, for a half-integral weight modular form to be in the Kohnen's +-subspace.

2. Multiplication by Theta series

In this section, we show that the Fourier coefficients of a half-integral weight modular form f are algebraic, if the Fourier coefficients of $f\Theta$ are algebraic.

Let $k > 1$ be an odd integer. Let f be a half-integral weight modular form of weight $k/2$, level N with $4 \mid N$, and an even Dirichlet character χ. In particular, $f \in M_{k/2}(\Gamma_1(N))$, therefore f has a Fourier expansion of the form $f(z) = \sum_{n=0}^{\infty} a_f(n)q^n$, where $q = e^{2\pi iz}$. Let $\Theta(z) = 1 + \sum_{n=1}^{\infty} b_\Theta(n)q^{n^2}$, with $b_\Theta(n) = 2$ for all n.

By [8], we know that $\Theta \in M_{1/2}(4, \chi_{\mathrm{triv}})$, where χ_{triv} stands for the trivial Dirichlet character. Define

$$g := f\Theta.$$

If $f \in M_{k/2}(N, \chi)$, then it is easy to see that $g \in M_{\frac{k+1}{2}}(N, \chi.\chi_{-1}^{\frac{k+1}{2}})$, where χ_{-1} is the non-trivial Dirichlet character modulo 4.

Suppose $g(z)$ has the q-expansion given by $g(z) = \sum_{n=0}^{\infty} c(n)q^n$. Since g is the product of f and Θ, we see that for all $n \in \mathbb{N}$, we have

$$c(n) = \sum_{x+y^2=n} a_f(x)b_\Theta(y).$$

In general, the product of two eigenforms need not be an eigenform, in particular $f\Theta$ need not be an eigenform. Hence, one cannot conclude that the Fourier coefficients of $f\Theta$ generate a number field. However, if we assume that Fourier coefficients of $f\Theta$ generate

2010 *Mathematics Subject Classification.* Primary 11F30, 11F37; Secondary 11F11.
Key words and phrases. Modular forms, half-integral weight, algebraicity, Sturm's bound.

a number field, then by a simple induction argument, we can show that the Fourier coefficients of f also generate the same number field.

Instead of proving the above claim for f and Θ, we prove it for the product of two general half-integral weight modular forms.

Proposition 2.1. *Let $f(z) = \sum_{n=n_0}^{\infty} a_f(n)q^n$, $h(z) = \sum_{n=n_1}^{\infty} b_h(n)q^n$ be two half-integral weight modular forms such that $a_f(n_0) \neq 0$, $b_h(n_1) \neq 0$. Suppose that the Fourier coefficients of the product fh belong to a number field K. If $a_f(i) \in K$ for all $i \geq n_0$, then $b_h(i) \in K$ for all $i \geq n_1$.*

Proof. We prove the proposition by induction. Suppose

$$fh = \sum_{n=n_0+n_1}^{\infty} c_n q^n$$

is the q-expansion of fh. For simplicity, we write a_n for $a_f(n)$ and b_n for $b_h(n)$.

Look at the first non-zero coefficient of the product fh, i.e., $n_0 + n_1$-th term of fh, which is $c_{n_0+n_1} = a_{n_0} b_{n_1} \in K$. Since $a_{n_0} \in K^*$, we get that $b_{n_1} \in K$.

Now, look at the $n_0 + n_1 + 1$-th term of the product fh, which is $c_{n_0+n_1+1} = a_{n_0} b_{n_1+1} + a_{n_0+1} b_{n_1} \in K$. Since $a_{n_0}, a_{n_0+1}, b_{n_1} \in K$, we get that $b_{n_1+1} \in K$.

Now, we assume that $b_{n_1}, b_{n_1+1}, \ldots, b_{n_1+r-1} \in K$ and show that $b_{n_1+r} \in K$. We can write

$$c_{n_0+n_1+r} = a_{n_0} b_{n_1+r} + a_{n_0+1} b_{n_1+r-1} + \ldots + a_{n_0+r} b_{n_1} \in K.$$

Since $b_{n_1}, b_{n_1+1}, \ldots, b_{n_1+r-1} \in K$ and $a_{n_0}, a_{n_0+1}, \ldots, a_{n_0+r} \in K$, we see that b_{n_1+r} also belongs to K. This proves the proposition. $\square$

Now, coming back to the pair (f, Θ), we have the following:

Corollary 2.2. *If the coefficients $c_n (n \in \mathbb{N})$ of $g = f\Theta$ belong to a number field K, then the Fourier coefficients $a_f(n)$'s are also algebraic. Further, $a_f(n) \in K$, $\forall n$.*

We shall illustrate the above proposition with an example.

Example 1. Take $k = 7$, $N = 8$. Using `MAGMA`[2], we see that $S_{7/2}(8, \chi_{\text{triv}})$ is one-dimensional and is spanned by

$$f = q - 2q^2 - 4q^5 + 12q^6 - 3q^9 - 20q^{10} + O(q^{12}).$$

Now consider the integral weight modular form $g := f\Theta$. This is an element of $S_4(8, \chi_{\text{triv}})$, which is of dimension 1. Moreover, $S_4^{\text{new}}(8, \chi_{\text{triv}})$ is also of dimension 1. By comparing the Fourier coefficient of q in $f\Theta$, we see that

$$S_4^{\text{new}}(8, \chi_{\text{triv}}) = \langle g \rangle.$$

By Proposition 2.1, we see that the Fourier coefficients of f generate a number field.

Remark 2.3. The same argument, as above, should go through, if you replace Θ by $\Theta(\psi, 0, z)$ for ψ an even Dirichlet character.

Let d be a positive integer and $V(d)$ be the usual shift operator. If $f \in M_{k/2}(N, \chi)$ then $V(d)f \in M_{k/2}(Nd, \left(\frac{4d}{\cdot}\right)\chi)$. Arguing as in the above proposition, one can prove:

Lemma 2.4. *Suppose $g(z) := f(z)[V(d)\Theta(z)]$. If the Fourier coefficients of $g(z)$ belongs to a number field K, then so are the Fourier coefficients of $f(z)$.*

3. Main result

In this section, we show that if the Fourier coefficients of a half-integral weight modular form are algebraic up to the Sturm's bound (for half-integral weight modular forms), which is specified in terms of the level and weight of the corresponding modular form, then so are all others.

Let k, N be positive integers with k odd and $4 \mid N$. Let χ be an even Dirichlet character of modulus N. In his thesis Basmaji [1] gave an algorithm for computing a basis for the space of half-integral weight modular forms of level divisible by 16. The main idea of the algorithm is to use theta series $\Theta = \sum_{n=-\infty}^{\infty} q^{n^2}$, $\Theta_1 = \frac{\Theta - V(4)\Theta}{2}$ and the following embedding,

$$\varphi : S_{k/2}(N, \chi) \to S \times S, \qquad f \mapsto (f\Theta, f\Theta_1),$$

where $S = S_{\frac{k+1}{2}}\left(N,\ \chi \cdot \chi_{-1}^{\frac{k+1}{2}}\right)$ and V is the usual shift operator. This idea was later generalized by Steve Donnelly for levels divisible by 4 by using different theta-multipliers.

We will be requiring the following lemma, which is an analogue of Sturm's Theorem [9, page 276] for half-integral weight forms.

Lemma 3.1. *Let $f = \sum_{n=0}^{\infty} a_f(n) q^n \in M_{k/2}(N, \chi)$ be a half-integral weight modular form. If $a_f(n) = 0$ for $n \leq \frac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$, then $f = 0$.*

Proof. Set $B := \frac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$. Since $a_f(n) = 0$ for $n \leq B$, the Fourier expansion of f at ∞ can be written as

$$f = q^{B+1}(a_f(B+1) + a_f(B+2)q + \cdots).$$

Let s be the order of the Dirichlet character χ. Then, it is easy to see that $f^{4s} \in M_{2ks}(\Gamma_0(N), \chi_{\mathrm{triv}})$ is an integral weight modular form. Clearly, the Fourier expansion of f^{4s} at ∞ looks like

$$f^{4s} = q^{4s(B+1)} \left(\sum_{n=0}^{\infty} c_n q^n \right),$$

where c_n is in terms of $a_f(i)$ for $i \leq n$.

Since the Fourier coefficients of f^{4s} are zero up to $4sB = \frac{2ks}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$, by applying Sturm's theorem [9, page 276] to f^{4s}, we get that $f^{4s} = 0$. This implies that, $f = 0$, which proves the lemma. $\square$

Our question is in the flavor of Sturm's work [9] on determining modular forms by their congruences by checking only up to a finite number.

Question 3.2. Can we determine the algebraicity of all the Fourier coefficients of f only by looking at its first "few" Fourier coefficients?

The following theorem gives a positive answer to the above question. Set $B_k(N) := \frac{k}{24} \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$.

Theorem 3.3. *Let $f = \sum_{n=1}^{\infty} a_f(n) q^n \in S_{k/2}(N, \chi)$ be a non-zero half-integral weight modular form. Suppose $a_f(m)$'s are algebraic for all $1 \leq m \leq B_k(N)$, then all other Fourier coefficients are algebraic. Moreover, there exists a number field K_f such that $a_f(n) \in K_f$, for all n.*

Proof. By Basmaji's algorithm [1], we can construct a basis for the space of cusp forms $S_{k/2}(N, \chi)$ such that the basis elements have Fourier coefficients defined over the number field generated by χ. Let $f_1, f_2, \ldots, f_r$ denote such a basis of $S_{k/2}(N, \chi)$. For each i,

suppose $f_i := \sum_{n=1}^{\infty} a_i(n)q^n$ is the q-expansion. Write $f = \sum_{i=1}^{r} \lambda_i f_i$ where $\lambda_i \in \mathbb{C}$. Hence, we have the following system of linear equations given by

$$\begin{bmatrix} a_1(1) & a_2(1) & \cdots & a_r(1) \\ \vdots & \vdots & \cdots & \vdots \\ a_1(m) & a_2(m) & \cdots & a_r(m) \\ \vdots & \vdots & \cdots & \vdots \\ a_1(n) & a_2(n) & \cdots & a_r(n) \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_r \end{bmatrix} = \begin{bmatrix} a_f(1) \\ \vdots \\ a_f(m) \\ \vdots \\ a_f(n) \\ \vdots \end{bmatrix} \tag{3.1}$$

For simplicity, let B denotes $B_k(N)$. Now from the above matrix, we consider first B rows to form the following $B \times r$-matrix, where $B \geq r$:

$$A := \begin{bmatrix} a_1(1) & a_2(1) & \cdots & a_r(1) \\ \vdots & \vdots & \cdots & \vdots \\ a_1(m) & a_2(m) & \cdots & a_r(m) \\ \vdots & \vdots & \cdots & \vdots \\ a_1(B) & a_2(B) & \cdots & a_r(B) \end{bmatrix} \tag{3.2}$$

We want to show that the rank of A is r. Suppose not. Then, there exists $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$, not all zero, such that

$$\sum_{i=1}^{r} \alpha_i a_i(j) = 0, \tag{3.3}$$

for each $j = 1, \ldots, B$. Now, consider the half-integral weight modular form

$$h := \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_r f_r.$$

Now, by (3.3), the first B coefficients of h are zero. By Lemma 3.1, we see that the modular form h is identically zero. Therefore, we have

$$\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_r f_r = 0.$$

Since $f_1, \ldots, f_r$ are linearly independent, we see that all α_i's have to be zero, which is a contradiction. Therefore, the matrix A has rank r. Let C be the $r \times r$ submatrix of A with full rank r. Now, consider the following system of linear equations (formed from the (3.1))

$$C \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_r \end{bmatrix} = \begin{bmatrix} a_f(i_1) \\ a_f(i_2) \\ \vdots \\ a_f(i_r) \end{bmatrix} \tag{3.4}$$

for some distinct $1 \leq i_1 < i_2 < \cdots < i_r \leq B$. Since C is invertible, we see that λ_i's can be expressed as an algebraic linear combination of $a_f(i_j)$ for $j = 1, \ldots, r$. Therefore, λ_i's are algebraic, hence all the Fourier coefficients of f are algebraic.

Now, take K_f to the number field generated by $\lambda_1, \ldots, \lambda_n$ and the values of χ. Since $f = \sum_{i=1}^{n} \lambda_i f_i$, we see that $a_f(n) \in K_f$ for all n. □

Remark 3.4. Using the same method one can see that the above theorem also holds for modular forms.

One could also use the idea of multiplying the half-integral weight modular form f by theta series Θ (as in Section 2) and argue in the integral weight case setting (cf. [5, Lemma 3.1]). In this approach, one may have to check the algebraicity of the Fourier coefficients of $f\Theta$ up to $\frac{k+1}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ to conclude that the all other coefficients of $f\Theta$ are algebraic, and hence the coefficients of f as well. However, if we work completely

in the half-integral weight setting, it is sufficient to check the algebraicity of the Fourier coefficients of f up to $B := \frac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ (in particular, this implies that only the first B coefficients of $f\Theta$ are algebraic). Hence, we chose to work in the half-integral weight setting which gives a bit better bound.

We predict that for an eigenform f such a bound for algebraicity can be sharpened by using Waldspurger's results [10]. We hope to come back to this in the future.

4. When $N/4$ is odd and square-free

Let k, N be positive integers with $k \geq 3$ odd and $4 \mid N$. Let χ be an even quadratic Dirichlet character of modulus N.

Let $F = \sum_{n=1}^{\infty} A(n)q^n$ be a newform of weight $k-1$, level $N/4$ odd and square-free with trivial nebentypus. Let f be an element of $S_{k/2}(N,\chi,F)$ (for the definition, see §4.1). In this case, checking the algebraicity of the Fourier coefficients of f becomes practically effective.

4.1. **An algebraic basis for $S_{k/2}(N,\chi,F)$:** In this section, we shall recall some basic definitions and results.

Let $S'_{k/2}(N,\chi)$ be the orthogonal complement of the subspace of $S_{k/2}(N,\chi)$ spanned by single-variable theta series with respect to the Petersson inner product. Note that for $k \geq 5$, we have $S'_{k/2}(N,\chi) = S_{k/2}(N,\chi)$.

Let $N' = N/2$. For $M \mid N'$ such that $\mathrm{Cond}(\chi^2) \mid M$ and a newform $F \in S^{\mathrm{new}}_{k-1}(M,\chi^2)$ Shimura defines

$$S_{k/2}(N,\chi,F) = \{f \in S'_{k/2}(N,\chi) : T_{p^2}(f) = \lambda_p^F f \text{ for almost all } p \nmid N\};$$

here $T_p(F) = \lambda_p^F F$. In [6, Corollary 5.2] the second author gives an algorithm for computing these subspaces. We will need the following proposition in the next section.

Proposition 4.1. *Let F be a newform in $S^{\mathrm{new}}_{k-1}(M,\chi^2)$ with level M dividing N'. Then there exists a basis of $S_{k/2}(N,\chi,F)$ defined over the number field generated by the Fourier coefficients of F and by χ.*

Proof. By Basmaji's algorithm [1], one can construct a basis for the space of cusp forms $S_{k/2}(N,\chi)$ defined over the number field generated by χ. Let $f_1, f_2, \dots f_r$ be such a basis and let $f_i = \sum_{n=1}^{\infty} a_i(n)q^n$. Let $p_1 < p_2 < \cdots < p_s$ be the primes chosen as in [6, Corollary 5.2]. Let $T_{p_j^2} f_i = \sum_{n=1}^{\infty} b_{i,j}(n)q^n$, for $1 \leq i \leq r$ and $1 \leq j \leq s$. By the same corollary, we see that to construct a basis for $S_{k/2}(N,\chi,F)$, we need to determine a basis of the solution space for the simultaneous homogeneous system of linear equations given by

$$\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ b_{1,1}(n) - \lambda^F_{p_1} a_1(n) & b_{2,1}(n) - \lambda^F_{p_1} a_2(n) & \dots & b_{r,1}(n) - \lambda^F_{p_1} a_r(n) \\ \vdots & \vdots & \dots & \vdots \\ b_{1,s}(n) - \lambda^F_{p_s} a_1(n) & b_{2,s}(n) - \lambda^F_{p_s} a_2(n) & \dots & b_{r,s}(n) - \lambda^F_{p_s} a_r(n) \\ \vdots & \vdots & \dots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}.$$

Since the matrix of the system is defined over number field generated by F and χ, the proposition clearly follows. $\square$

4.2. **On determination of the algebraicity.** Let $F = \sum_{n=1}^{\infty} A(n)q^n$ be a newform of weight $k-1$ and level $N/4$ odd and square-free, with trivial nebentypus. Let f be an element of $S_{k/2}(N,\chi,F)$.

By Waldspurger [10, Théorème 1], we know that $S_{k/2}(N,\chi,F)$ is 2-dimensional. By Proposition 4.1, we know that there exists a basis f_1, f_2 of $S_{k/2}(N,\chi,F)$, which are defined over the field generated by the Fourier coefficients of F and the values of χ. In particular,

f_1, f_2 have algebraic Fourier coefficients. Now, we shall prove that, we can take f_1 to be a generator of $S^+_{k/2}(N,\chi,F)$.

By [3, Theorem 2], the Kohnen's $+$-subspace $S^+_{k/2}(N,\chi,F)$ is 1-dimensional. Suppose that $S^+_{k/2}(N,\chi,F) = \langle f_K\rangle$, where $f_K(z) = \sum_{n=1}^{\infty} a_K(n)q^n$ with $a_K(n) = 0$, unless $n \equiv 0, (-1)^{\frac{k-1}{2}} \pmod 4$.

Proposition 4.2. *The Fourier coefficients of f_K, up to a normalization, are algebraic.*

Proof. We know that the coefficients $a_K(n)$ and $A(n)$ of $f_K(z)$ and $F(z)$ are related by

$$A(m)a_K(|D|) = \sum_{\substack{d|m \\ (d,N)=1}} \left(\frac{D}{d}\right) d^{\frac{k-3}{2}} a_K\left(\frac{|D|m^2}{d^2}\right). \tag{4.1}$$

for all $n \in \mathbb{N}$ and all fundamental discriminant D with $(-1)^{\frac{k-1}{2}}D > 0$ (cf. [3, page 64]).

It follows that by multiplying f_K with $a_K(|D_0|)^{-1}$ where D_0 is a suitable fundamental discriminant with $(-1)^{\frac{k-1}{2}}D_0 > 0$ and $a_{f_K}(|D_0|) \neq 0$ one can assume that the Fourier coefficients of f_K lie in the number field $K_{f_K} := \mathbb{Q}(\{A(p)\}_{p\nmid N},\ \mu_2,\ \{\mu_p\}_{p|(N/4)})$; here μ_2 is the Hecke eigenvalue of f_K under the modified Hecke operator T_4^+ (see [3]), while μ_p for $p \mid (N/4)$ are the Hecke eigenvalues of f_K under the usual Hecke operators $T(p^2)$ on $S^+_{k/2}(N,\chi)$. Clearly, the values μ_p, for $p \mid N$, are algebraic. Hence, the form f_K has algebraic Fourier coefficients. Note that $K_{f_K} \subseteq K_F(\{\mu_p\}_{p|N})$ where K_F is the number field of F. Hence, the Fourier coefficients of f_K, up to a normalization, are algebraic. $\square$

By the above proposition, we can now choose the basis element f_1 to be equal to f_K. By a similar argument as in Theorem 3.3, we can prove the following result for $S_{k/2}(N,\chi,F)$.

Corollary 4.3. *Let $f = \sum_{n=1}^{\infty} a_f(n)q^n$ be an element of $S_{k/2}(N,\chi,F) = \langle f_1, f_2\rangle$, where $f_i(z) := \sum_{n=1}^{\infty} a_i(n)q^n$, with $a_i(*)$'s are algebraic. Choose $m_0 \in \mathbb{N}$ with $m_0 \not\equiv 0, (-1)^{\frac{k-1}{2}} \pmod 4$ such that $a_1(m_0) = 0$ and $a_2(m_0) \neq 0$. Choose $n_0 \in \mathbb{N}$ such that $a_1(n_0) \neq 0$.*

If $a_f(m_0)$, $a_f(n_0)$ are algebraic, then $a_f(n)$ are algebraic, for all n. Moreover, there exists a number field K_f such that $a_f(n) \in K_f$ for all n.

Remark 4.4. One can always find such an m_0, n_0 as in the above Corollary, since $f_2 \notin S^+_{k/2}(N,\chi,F)$ and $f_1 \neq 0$.

We illustrate the results in this section with a few examples.

Example 2. Let F be the newform in $S_2^{\text{new}}(91)$ given by the following Fourier expansion

$$F = q - 2q^2 + 2q^4 - 3q^5 - q^7 - 3q^9 + 6q^{10} - 6q^{11} + O(q^{12}).$$

We note that F is the newform corresponding to the rank 1 elliptic curve defined by $y^2 + y = x^3 + x$. By [6, Corollary 5.2], we can determine that the space $S_{3/2}(364, \chi_{\text{triv}}, F)$ is generated by

$$f_1 = q^3 - q^{12} + q^{35} - q^{40} + O(q^{50})$$
$$f_2 = q^{10} + q^{12} + q^{13} - q^{14} + q^{17} + q^{26} - 3q^{38} - q^{40} - 2q^{42} - 2q^{48} + O(q^{50}).$$

Here f_1 is in the Kohnen's $+$-subspace and f_2 does not belong to the Kohnen's $+$-subspace. Thus to see whether a given eigenform in $S_{3/2}(364, \chi_{\text{triv}}, F)$ has algebraic coefficients, we need to take $m_0 = 10$ and $n_0 = 3$ in Corollary 4.3.

Example 3. Let G be the newform in $S_4^{\text{new}}(13)$ given by

$$G = q + bq^2 + (-3b + 4)q^3 + (b - 4)q^4 + (b - 2)q^5 + O(q^6),$$

where minimal polynomial for b is given by $x^2 - x - 4$. Let χ_{13} be the quadratic Dirichlet character given by the Kronecker symbol $\left(\frac{13}{\cdot}\right)$. By [6, Corollary 5.2], we can determine that the space $S_{5/2}(52, \chi_{13}, G)$ is generated by

$$f_1 = q + (b+2)q^4 + 1/2(-5b-8)q^5 + O(q^6)$$
$$f_2 = q^2 + 1/19(-2b+14)q^3 + (-b+1)q^4 + 1/19(18b+26)q^5 + O(q^6).$$

Here f_1 is in the Kohnen's $+$-subspace while f_2 is not in the Kohnen's $+$-subspace. Thus for a given eigenform in $S_{5/2}(52, \chi_{13}, F)$ to have algebraic coefficients, we need to just check for $m_0 = 2$ and $n_0 = 1$ in Corollary 4.3.

5. Sturm's bound for forms in the Kohnen's $+$-subspace

In this section, we will give a necessary and sufficient condition for a half-integral weight modular form to be in the Kohnen's $+$-subspace. Recall that $S^+_{k/2}(N, \chi)$ denote the subspace of forms whose Fourier coefficients $a_n(f)(n \in \mathbb{N})$ are zero for $n \equiv 2, (-1)^{\frac{k+1}{2}} \pmod 4$. Set $B'_k(N) := \frac{k}{24} \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]$.

Proposition 5.1. *Suppose $f = \sum_{n=1}^{\infty} a_f(n)q^n \in S_{k/2}(N, \chi)$. Then, $f \in S^+_{k/2}(N, \chi)$ if and only if $a_f(n) = 0$ for all $n \equiv 2, (-1)^{\frac{k+1}{2}} \pmod 4$ with $1 \le n \le B'_k(N)$.*

Proof. For simplicity, let B denotes $B'_k(N)$. The necessary condition is clear, by definition.

Suppose that $a_f(n) = 0$ for all $1 \le n \le B$ with $n \equiv 2, (-1)^{\frac{k+1}{2}} \pmod 4$. Since $a_f(n) = 0$ for all $1 \le n \le B$ with $n \equiv (-1)^{\frac{k+1}{2}} \pmod 4$, the Sturm's bound for the coefficients of half-integral weight forms in arithmetic progressions [7, Section 7] implies that $a_f(n) = 0$ for all n with $n \equiv (-1)^{\frac{k+1}{2}} \pmod 4$.

Now, it is enough show that $a_f(n) = 0$ for all $1 \le n \le B$ with $n \equiv 2 \pmod 4$, implies that $a_f(n) = 0$ for all $n \in \mathbb{N}$ with $n \equiv 2 \pmod 4$. Unfortunately, we cannot apply the above result in this situation, because $(2, 4) \neq 1$. However, we can get around to prove the required statement.

Applying U and V-operators to f (cf. [8]), we obtain that

$$V(4) \circ U(4)f(z) = \sum_{n=1}^{\infty} a_f(4n)q^{4n} \in S_{k/2}(4N, \chi),$$

and

$$V(2) \circ U(2)f(z) := \sum_{n=1}^{\infty} a_f(2n)q^{2n} \in S_{k/2}(2N, \chi).$$

Take

$$g(z) = V(2) \circ U(2)f(z) - V(4) \circ U(4)f(z) = \sum_{n=1}^{\infty} a_f(4n-2)q^{4n-2} \in S_{k/2}(4N, \chi).$$

Since $a_f(n) = 0$ for all $1 \le n \le B$ with $n \equiv 2 \pmod 4$, we see that the first B coefficients of $g(z)$ are zero. Then, by Lemma 3.1, we obtain that $g(z) = 0$. Hence, $a_f(n) = 0$ for all $n \equiv 2 \pmod 4$. This proves the proposition. $\square$

It follows from the above proposition that in Corollary 4.3, we can choose m_0, $n_0 \le B'_k(N)$.

References

[1] Basmaji, Jacques. Ein Algorithmus zur Berechnung von Hecke-Operatoren und Anwendungen auf modulare Kurven. Ph. D. Dissertation, Universität Gesamthochschule Essen, März 1996.

[2] Bosma, W., Cannon J., and Playoust, C. The Magma Algebra System I: The User Language. J. Symb. Comp. 24 (1997), 235–265. (See also `http://magma.maths.usyd.edu.au/magma/`)

[3] Kohnen, Winfried. Newforms of half-integral weight. J. Reine Angew. Math. 333 (1982), 32–72.

[4] Kohnen, Winfried. On Hecke eigenforms of half-integral weight. Math. Ann. 293 (1992), no. 3, 427–431.

[5] Kohnen, Winfried and Mason, Geoffrey. On the canonical decomposition of generalized modular functions. Proc. Amer. Math. Soc. 140 (2012), no. 4, 1125–1132.

[6] Purkait, Soma. On Shimura's Decomposition. To appear in Int. J. Number Theory.

[7] Purkait, Soma. Explicit applications of Waldspurger's Theorem. To appear in LMS J. Comput. Math.

[8] Shimura, Goro. On Modular forms of half integral weight. Annals of Math. 97 (1973), 440–481.

[9] Sturm, Jacob. On the congruence of modular forms. Number theory (New York, 1984-1985), 275–280, Lecture Notes in Math., 1240, Springer, Berlin, 1987.

[10] Waldspurger, J. -L. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. J. Math. Pures Appl. (9) 60 (1981), no. 4, 375–484.

E-mail address: `narasimha.kumar@iith.ac.in`

Department of Mathematics, Indian Institute of Technology Hyderabad, Ordnance Factory Estate, Yeddumailaram 502205, Andhra Pradesh, INDIA.

E-mail address: `Soma.Purkait@warwick.ac.uk`

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom.